Ranking Alternatives From Comparison Data

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Ranking Alternatives

Goal: use voter data to rank the best alternatives
Voters: indicate a preference for certain alternatives

Examples:

Goal	Alternatives	Voters
Tennis Player Standings	Players	Matches
Rank Netflix Shows	Shows	Users
Web Ranking	Webpages	Links

Example: Tennis Matches

Winner	Loser
Anne	Bob
Anne	Carl
Anne	Carl
Anne	Carl
Bob	Carl
Bob	Carl
Carl	Bob
Bob	Dan
Dan	Bob
Carl	Dan

Example: Tennis Matches

Winner	Loser
Anne	Bob
Anne	Carl
Anne	Carl
Anne	Carl
Bob	Carl
Bob	Carl
Carl	Bob
Bob	Dan
Dan	Bob
Carl	Dan

$$Y = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & -1 & -3 & 0 \\ 1 & 0 & -1 & 0 \\ 3 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$w = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 3 & 0 \\ 1 & 0 & 3 & 2 \\ 3 & 3 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

$$\bar{Y} = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & -\frac{1}{3} & 0 \\ 1 & \frac{1}{3} & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Making a Guess

Guess a Rough Ranking:

$$s = \begin{bmatrix} \overset{A}{3} & \overset{B}{1} & \overset{C}{0} & \overset{D}{1} \end{bmatrix}^T$$

Expected results:

$$match_{ab} = s_b - s_a = 1 - 3 = \text{-2}$$

Can think of as the gradient, because it captures the difference

$$\mathrm{grad}(s) = \begin{bmatrix} 0 & \text{-2} & \text{-3} & \text{-2} \\ 2 & 0 & \text{-1} & 0 \\ 3 & 1 & 0 & 1 \\ 2 & 0 & \text{-1} & 0 \end{bmatrix}$$

How Good Was Our Guess?

$$\bar{Y} = \begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & -\frac{1}{3} & 0 \\ 1 & \frac{1}{3} & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \quad \mathrm{grad}(s) = \begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} 0 & 1 & 2 & 2 \\ -1 & 0 & \frac{2}{3} & 0 \\ -2 & -\frac{2}{3} & 0 & -2 \\ -2 & 0 & 2 & 0 \end{bmatrix} \end{array}$$

$$E = \bar{Y} - \mathrm{grad}(s) = \begin{bmatrix} 0 & 1 & 2 & 2 \\ -1 & 0 & \frac{2}{3} & 0 \\ -2 & -\frac{2}{3} & 0 & -2 \\ -2 & 0 & 2 & 0 \end{bmatrix}$$

Let's take the Frobenius norm! $||E||_2 = (\Sigma_{i,j} E_{ij}^2)^{0.5}$

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Let's take the Frobenius norm! $||E||_2 = (\Sigma_{i,j} E_{ij}^2)^{0.5}$
... but we need to account for the weight

$$w = \begin{bmatrix} 0 & 1 & 3 & 0 \\ 1 & 0 & 3 & 2 \\ 3 & 3 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \quad ||E||_{2,w} = (\Sigma_{i,j} w_{ij} E_{ij}^2)^{0.5} \approxeq 8.37$$

Quantifying the Error

Two Measures:

1. Error: $||\bar{Y} - \mathrm{grad}(s)||_{2,W}$
2. Relative Error: $\frac{||\bar{Y} - \mathrm{grad}(s)||_{2,w}}{||\bar{Y}||_{2,w}}$

Error	8.37
Relative Error	2.56

Computing Best Solution (Slide 1 of 3)

Want to solve for s in $\min ||\bar{Y} - \text{grad}(s)||_{2,w}$

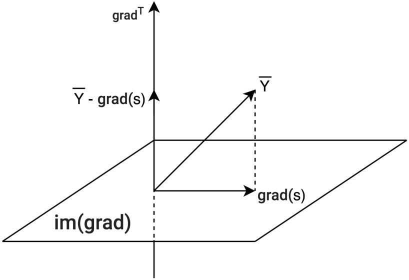
Solution: use linear algebra!

So how do represent the gradient as a matrix?

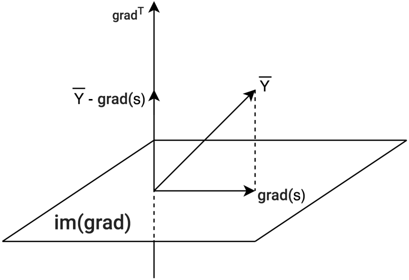
$$\text{Want:} \quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} a \text{ vs. } b \\ a \text{ vs. } c \\ a \text{ vs. } d \\ b \text{ vs. } c \\ b \text{ vs. } d \\ c \text{ vs. } d \end{bmatrix}$$

$$M_{\text{grad}} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Computing Best Solution (Slide 2 of 3)



Computing Best Solution (Slide 2 of 3)



$$\bar{Y} - \mathrm{grad}(s) \perp \mathrm{im}(\mathrm{grad}) \Leftrightarrow \langle \bar{Y} - \mathrm{grad}(s), \mathrm{grad}(x) \rangle = 0 \quad \forall x$$
$$\Leftrightarrow \langle \mathrm{grad}^*(\bar{Y} - \mathrm{grad}(s)), x \rangle = 0 \quad \forall x$$
$$\Leftrightarrow \mathrm{grad}^*(\bar{Y} - \mathrm{grad}(s)) = 0$$

We solve to get: $s = (\mathrm{grad}^*\mathrm{grad})^{-1}\mathrm{grad}^* y$

Computing Best Solution (Slide 3 of 3)

Want matrix representation for grad^*
Normally, we'd have $M_{\mathrm{grad}^*} = M_{\mathrm{grad}}^T$.

$$M_{\mathrm{grad}}^T = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \leftarrow \text{multiply by} - \begin{bmatrix} ab \\ ac \\ ad \\ bc \\ bd \\ cd \end{bmatrix}$$

Computing Best Solution (Slide 3 of 3)

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...but must acct for weighted inner product: $\langle \mathrm{grad} f, \bar{Y} \rangle_w = \langle f, \mathrm{grad}^* \bar{Y} \rangle$

$$M_{\mathrm{grad}^*} = M_{\mathrm{grad}}^T \mathrm{diag}(w) = \begin{bmatrix} -1 & -3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -3 & -2 & 0 \\ 0 & 3 & 0 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 2 & 1 \end{bmatrix}$$

Finally, the Solution

$$M_{\text{grad}} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad M_{\text{grad}^*} = \begin{bmatrix} 1 & 3 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 3 & 2 & 0 \\ 0 & -3 & 0 & -3 & 0 & 1 \\ 0 & -3 & 0 & -3 & 0 & 1 \\ 0 & 0 & 0 & 0 & -2 & -1 \end{bmatrix}$$

$$\begin{aligned} s &= (\text{grad}^*\text{grad})^{-1}\text{grad}^* \bar{Y} \\ &= \begin{bmatrix} 0.81 & -0.13 & -0.203 & -0.486 \end{bmatrix} \\ & \quad\;\; \text{A} \qquad\;\; \text{B} \qquad\;\; \text{C} \qquad\;\; \text{D} \end{aligned}$$

How Good is the Solution?

$$\begin{matrix} A & B & C & D \\ \end{matrix}$$
$$\begin{bmatrix} 0.81 & -0.13 & -0.203 & -0.486 \end{bmatrix}$$

$$\text{grad}(s) = \begin{bmatrix} 0 & -0.94 & -1.02 & -0.28 \\ 0.94 & 0 & -0.08 & -0.36 \\ 1.02 & 0.08 & 0 & -0.28 \\ 0.28 & 0.36 & 0.28 & 0 \end{bmatrix} \qquad \bar{Y} = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & -\frac{1}{3} & 0 \\ 1 & \frac{1}{3} & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Measure	Formula	Guess	Solution
Error	$\|\|\bar{Y} - \text{grad}(s)\|\|_{2,W}$	8.37	2.95
Relative Error	$\frac{\|\|\bar{Y} - \text{grad}(s)\|\|_{2,w}}{\|\|\bar{Y}\|\|_{2,w}}$	2.56	0.90

Ranking Real Data

Denote *Rankability* $= \left(\frac{||\text{grad}(\boldsymbol{s})||_{2,w}}{||\bar{Y}||_{2,w}}\right)^2$

- Note: *Rankability* $\in [0, 1]$

Ranking Real Data

Denote *Rankability* $= \left(\frac{||\text{grad}(s)||_{2,w}}{||\bar{Y}||_{2,w}}\right)^2$

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1. 2017 Major Tennis Tournaments
   - 453 matches, 50 players
     - Not many matches for each player
   - *Rankability* $= 0.37$

Ranking Real Data

Denote *Rankability* $= \left(\frac{||\text{grad}(s)||_{2,w}}{||\bar{Y}||_{2,w}} \right)^2$

- Note: *Rankability* $\in [0, 1]$

1. 2017 Major Tennis Tournaments
   - 453 matches, 50 players
     - Not many matches for each player
   - *Rankability* $= 0.37$
2. Major Golf Tournaments in 2018
   - 4 tournaments, 50 players
     - Every tournament compares all 50 players
     - So $4 * \binom{50}{2} = 4900$ voters
   - *Rankability* $= 0.63$
3. 9-Round Chess Tournament
   - 119 matches, 50 players
   - *Rankability* $= 0.45$

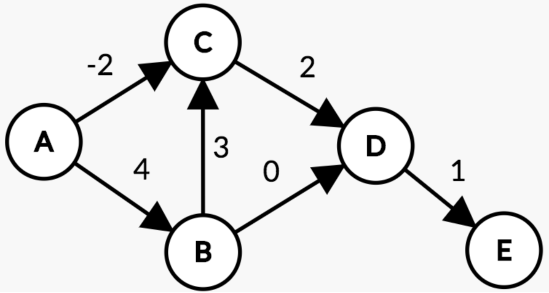
Data Where Ranking is Impossible

Winner	Loser
Anne	Bob
Bob	Carl
Carl	Anne

- This data is circular
- No ranking works

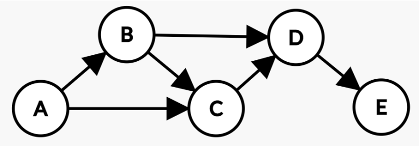
$$\bar{Y} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \qquad s = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Rankability = \left(\frac{||\mathrm{grad}(s)||_{2,w}}{||\bar{Y}||_{2,w}} \right)^2 = 0.0$$


C
-2
2
A
4
3
0
D
1
B
E

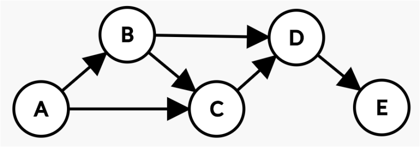
High-Level Overview

Perfect ranking $\implies$ directed acyclic graph

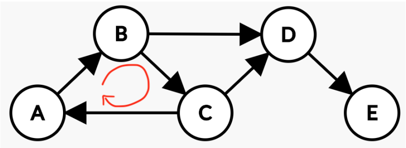


High-Level Overview

Perfect ranking $\implies$ directed acyclic graph



Otherwise, there are circular flows

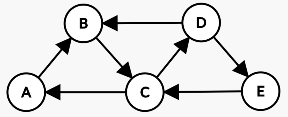


Two ways to think of circular flows:

1. (Good) Like an electric current
2. (Bad) Can't be ranked

Two Types of Flows

(1) Triangular inconsistencies and linear combinations thereof



Call these "curl flows"

Two Types of Flows

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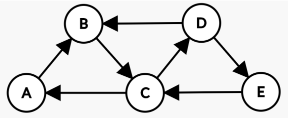
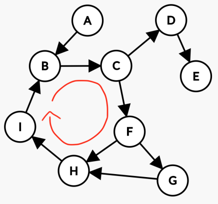


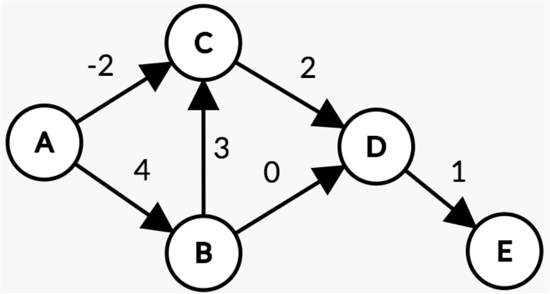
Call these "curl flows"

(2) Other flows that can't be reduced to triangular inconsistencies

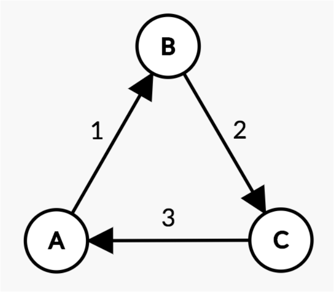


These correspond to "harmonic flows"

Computing the Curl Flow - An Example



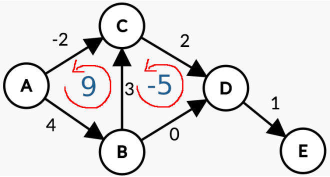
Computing the Curl



$$\begin{aligned}\mathrm{curl}(abc) &= ab + bc + ca \\ &= 1 + 2 + 3 \\ &= 6\end{aligned}$$

Note: $ca = -ac$, so $ab + bc + ca = ab + bc - ac$.

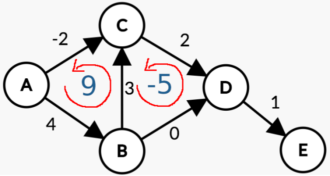
Computing the Curl of All Triangles



Want to compute the curl for each triangle

- $\text{curl}(abc) = ab - ac + bc$
- $\text{curl}(bcd) = bc - bd + cd$

Computing the Curl of All Triangles



Want to compute the curl for each triangle

- $\mathrm{curl}(abc) = ab - ac + bc$
- $\mathrm{curl}(bcd) = bc - bd + cd$

$$\begin{matrix} \text{abc} \\ \text{bcd} \end{matrix} \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 3 \\ 0 \\ 2 \\ 1 \end{bmatrix} \begin{matrix} \text{ab} \\ \text{ac} \\ \text{bc} \\ \text{bd} \\ \text{cd} \\ \text{de} \end{matrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$$

Denote the first matrix M_{curl}, and the second matrix $\bar{Y}$

What is M_{curl^*}?

First guess: $M_{\mathrm{curl}^*} = M_{\mathrm{curl}}^T$

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 1 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

...but must account for weighted inner product: $\langle \mathrm{curl} a, b \rangle = \langle a, \mathrm{curl}^* b \rangle_w$

So, divide the rows by edge weights:

$$\text{(Weights:)} \begin{bmatrix} 4 \\ 2 \\ 3 \\ 2 \\ 2 \\ 1 \end{bmatrix} \qquad M_{\mathrm{curl}^*} = \mathrm{diag}(w)^{-1} M_{\mathrm{curl}^T} = \begin{bmatrix} 0.25 & 0 \\ -0.5 & 0 \\ 0.33 & 0.33 \\ 0 & -0.5 \\ 0 & 0.5 \\ 0 & 0 \end{bmatrix}$$

Make a Guess (Slide 1 of 2)

- Before: Approximate $\bar{Y}$ with a ranking
- Now: Approximate $\bar{Y}$ with triangle values

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Guess:

$$\begin{bmatrix} abc \\ bcd \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

Resulting edge values:

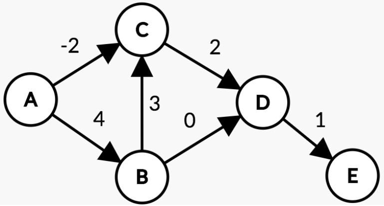
$$M_{\mathrm{curl}^*} guess = \begin{bmatrix} 0.25 & 0 \\ -0.5 & 0 \\ 0.33 & 0.33 \\ 0 & -0.5 \\ 0 & 0.5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.5 \\ -3 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Make a Guess (Slide 2 of 2)

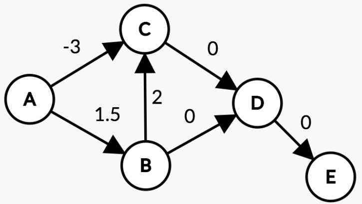
Guess:

$$\begin{bmatrix} abc \\ bcd \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$\bar{Y}$ (actual):



$\mathrm{curl}^* guess$:



Measure	Formula	Value
Error	$\lvert\lvert \bar{Y} - \mathrm{grad}(s) \rvert\rvert_{2,W}$	8.83
Relative Error	$\frac{\lvert\lvert \bar{Y} - \mathrm{grad}(s) \rvert\rvert_{2,w}}{\lvert\lvert \bar{Y} \rvert\rvert_{2,w}}$	0.60

The Least-Squares Solution

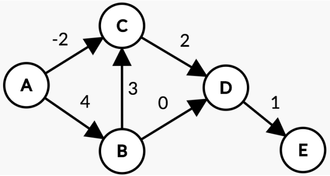
Curl Values:

$$\begin{bmatrix} abc \\ bcd \end{bmatrix} = \begin{bmatrix} 7.75 \\ 1.81 \end{bmatrix}$$

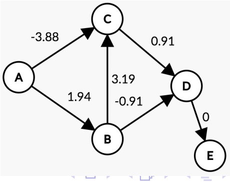
Resulting edge values:

$$M_{\mathrm{curl}^*}\,solution = \begin{bmatrix} 0.25 & 0 \\ \text{-}0.5 & 0 \\ 0.33 & 0.33 \\ 0 & \text{-}0.5 \\ 0 & 0.5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 7.75 \\ 1.81 \end{bmatrix} = \begin{bmatrix} 1.94 \\ \text{-}3.88 \\ 3.19 \\ \text{-}0.91 \\ 0.91 \\ 0 \end{bmatrix}$$

$\bar{Y}$ (actual):



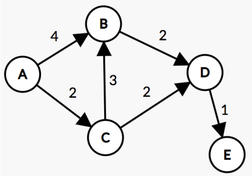
Curl Flow:



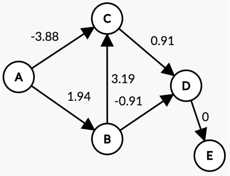
Computing the Error

Measure	Formula	Guess	Solution
Error	$\|\|\bar{Y} - \mathrm{grad}(s)\|\|_{2,W}$	8.83	7.63
Relative Error	$\frac{\|\|\bar{Y} - \mathrm{grad}(s)\|\|_{2,w}}{\|\|\bar{Y}\|\|_{2,w}}$	0.60	0.52

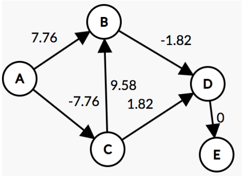
How to Think About the Solution



(a) Weights = Conductance



(b) Edge = Potential Difference



(a)∗(b) = Current

The Harmonic Flow

So far, we've captured:

- Portion of $\bar{Y}$ accounted for by a ranking
- Portion of $\bar{Y}$ accounted for by a triangular flow

What remains:

- Portion of $\bar{Y}$ accounted for by non-triangular flow

We call this the harmonic flow.

The Harmonic Flow

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Harmonic flow must be:

- Divergence-free (so that it's a flow)
- Curl-free (we already captured local flow)

Formally, the space of all harmonic flows is:

$$\begin{aligned} S_H &= \ker(\mathrm{div}) \cap \ker(\mathrm{curl}) \\ &= \ker(\mathrm{grad}^*) \cap \ker(\mathrm{curl}) \end{aligned}$$

Computing the Best Harmonic Flow

Claim: $\ker(\mathrm{grad}^*) \cap \ker(\mathrm{curl}) = \ker(\mathrm{curl}^* \circ \mathrm{curl} + \mathrm{grad} \circ \mathrm{grad}^*)$

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$\subseteq$: Straightforward

Computing the Best Harmonic Flow

Claim: $\ker(\text{grad}^*) \cap \ker(\text{curl}) = \ker(\text{curl}^* \circ \text{curl} + \text{grad} \circ \text{grad}^*)$

$\subseteq$: Straightforward

$\supseteq$:
Suppose $x \in \ker(\text{curl}^* \circ \text{curl} + \text{grad} \circ \text{grad}^*)$.
Then

$$\begin{aligned} 0 &= \langle x, 0 \rangle \\ &= \langle x, (\text{curl}^* \circ \text{curl} + \text{grad} \circ \text{grad}^*)x \rangle \\ &= \langle x, (\text{curl}^* \circ \text{curl})x \rangle + \langle x, (\text{grad} \circ \text{grad}^*)x \rangle \\ &= \langle \text{curl}x, \text{curl}x \rangle + \langle \text{grad}^* x, \text{grad}^* x \rangle \\ &= ||\text{curl}x||^2 + ||\text{grad}^* x||^2 \end{aligned}$$

$\implies ||\text{curl}x|| = ||\text{grad}^* x|| = 0.$
$\implies x \in \ker(\text{curl})$ and $x \in \ker(\text{grad}^*)$
$\implies x \in \ker(\text{grad}^*) \cap \ker(\text{curl}) \quad \checkmark$

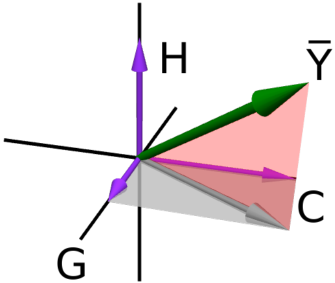
Flows are Perpendicular

We can now find:

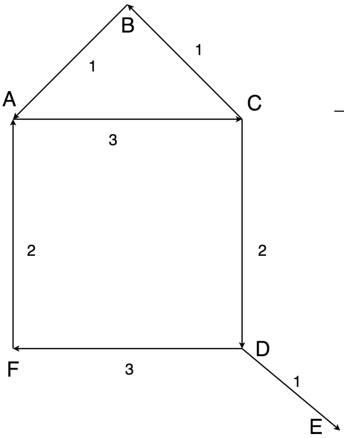
- G - Gradient part
- C - Curl flow
- H - Harmonic Flow

We'll see that:

- G, C, and H are perpendicular
- $G + C + H = \bar{Y}$



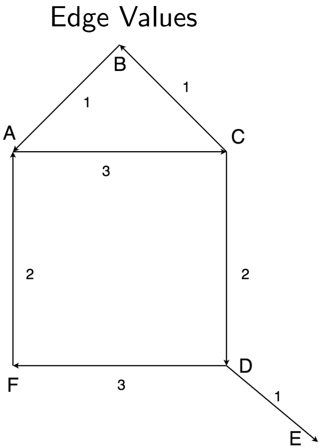
Overview by Example



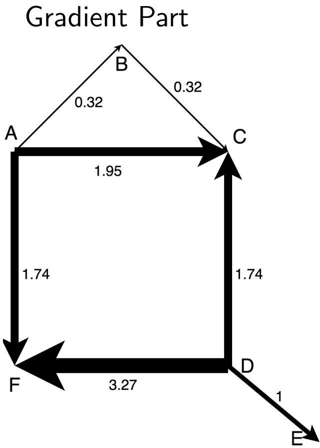
	Gradient	Curl Flow	Harmonic Flow
ab	0.32	-2.14	0.82
ac	0.65	0.71	1.64
af	0.87	0	-2.87
bc	-0.32	-2.14	0.82
cd	-0.87	0	2.87
de	1	0	0
df	1.09	0	1.91

Note: each row sums to the corresponding edge value!

Overview (Gradient)



	Gradient
ab	0.32
ac	0.65
af	0.87
bc	-0.32
cd	-0.87
de	1
df	1.09

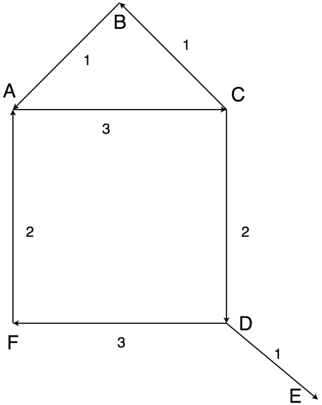


Notice:

- It's a directed acyclic graph (no flows)

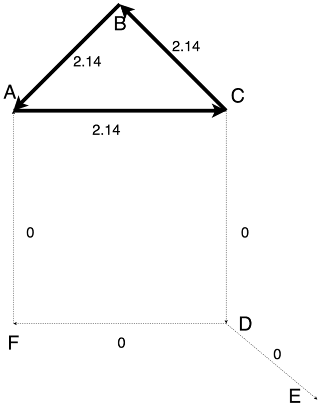
Overview (Curl Flow)

Edge Values



	Curl
ab	-2.14
ac	0.71
af	0
bc	-2.14
cd	0
de	0
df	0

Curl Flow

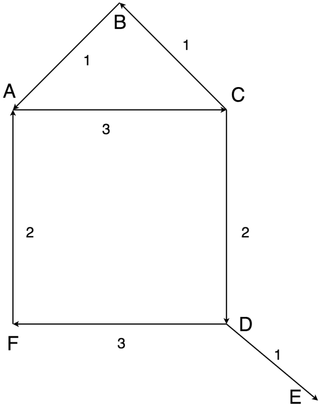


Notice:

- Only the edges in a triangle have a nonzero-values
- All edges in this triangle have the same value

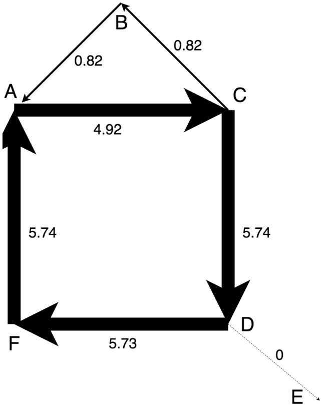
Overview (Harmonic Flow)

Edge Values



	Harm
ab	0.82
ac	1.64
af	-2.87
bc	0.82
cd	2.87
de	0
df	1.91

Harmonic Flow



Notice:

- The edges in the non-local loop dominate

Why the Components are Orthogonal

(1) Why G is orthogonal to C and H

$$\begin{aligned} a \in \mathrm{im}(\mathrm{grad})^{\perp} &\iff \langle \mathrm{grad}(f), a \rangle = 0 \quad \forall f \\ &\iff \langle f, \mathrm{grad}^* a \rangle = 0 \quad \forall f \\ &\iff \mathrm{grad}^* a = \mathrm{div}(a) = 0 \\ &\iff a \text{ is a flow} \qquad \checkmark \end{aligned}$$

Since C and H are flows, then: $C, H \perp G$.

Why the Components are Orthogonal

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Since C and H are flows, then: $C, H \perp G$.

(2) Why C is orthogonal to H

$$\begin{aligned} a \in \mathrm{im}(curl^*)^{\perp} &\iff \langle \mathrm{curl}^* A, a \rangle = 0 \quad \forall A \in \mathrm{im}(curl^*)^{\perp} \\ &\iff \langle A, \mathrm{curl}(a) \rangle = 0 \quad \forall A \\ &\iff \mathrm{curl}(a) = 0 \\ &\iff a \text{ is curl-free} \qquad \checkmark \end{aligned}$$

Since H is curl-free, then $H \perp G$.

Real Data - Revisited

For each flow $F \in \{G, C, H\}$, compute $\left(\frac{||F||_{2,w}}{||\bar{Y}||_{2,w}}\right)^2$

	Gradient	Curl Flow	Harmonic Flow
Tennis	0.36	0.64	0.001
Golf	0.63	0.37	0
Chess	0.45	0.04	0.51

Observations:

- Rankability: same as before
- Golf: no harmonic flow because all triangles filled in
- Tennis: also low harmonic flow

Comparing to a Random Baseline

Using actual data:

	Gradient	Curl Flow	Harmonic Flow
Tennis	0.36	0.64	0.001
Golf	0.63	0.37	0
Chess	0.45	0.04	0.51

After randomizing edges and edge values (preserving sparsity):

	Gradient	Curl Flow	Harmonic Flow
Tennis	0.21	0.58	0.21
Golf	0.06	0.94	0.0
Chess	0.40	0.000035	0.60

Observations:

- Randomized chess data had high gradient

Acknowledgements

- Ideas drawn from *Statistical ranking and combinatorial Hodge theory* by Xiaoye Jiang, Lek-Heng Lim, Yuan Yao, and Yinyu Ye
- Prof. De Silva for explanations and ideas for new directions